# Subdivision for Generating Quadrics

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**Abstract** Quadrics are of basic importance in Computer Graphics and Computer Aided Design. In this paper, we design a subdivision scheme based on the method suggested by G. Morin and J. Warren to generate conics and quadrics conveniently. Given the control polygon(poly-hedron), the corresponding ellipse(ellipsoid) can be generated. The hyperbolas and hyperboloids are generated based on the generation of ellipses and ellipsoids by a simple transformation. The method in this paper is much simpler and easier to apply than those given by Eugenia Montiel et al.

**Keywords** Subdivision surface, Quadrics, Geometric modeling

## 1 Introduction

Subdivision is an important method in geometric modeling, and it can be looked as a generalization of spline curves/surfaces under arbitrary geometric topology. Since the first subdivision scheme for surfaces-the Catmull-Clark scheme[3] appeared in 1970s, subdivision techniques have been developed rapidly in recent years, and have been widely used in Computer Graphics and CAGD for its simplicity and efficiency.

Although there are many subdivision schemes such as Loop scheme[4], Butterfly scheme[5] etc., these scheme can only generate a limit types of curves and surfaces. Even for some simple but of basic importance curves and surfaces such as conics and quadrics, these schemes are invalid. Sederberg et al.[6] described a subdivision scheme based on the extension of NURBS that is capable of representing sphere and cylinders. However, due to the fact that the rational parameterizations for these surfaces are non-uniform, the resulting subdivision scheme cannot represent these surfaces in their natural acr-length parameterization and the implementation of the scheme is complicated. Eugenia Montiel et al.[2] also presented a subdivision scheme to generate quadrics. The scheme is an non-uniform interpolatory subdivision scheme, and the basic idea of which is as follows: given the initial points on the ellipse as the control points, the points on the new level are computed by the two adjacent control points. In order to compute the new level of points, it need to compute the subdivision mask according to each new control points. The method to generate ellipses by the scheme is much more complex.

In this paper, we propose a new subdivision scheme based on the methods developed by G. Morin et al.[1] to generate conics and quadrics conveniently. Given the control polygon(polyhedron), the corresponding ellipse(ellipsoid) is generated. The hyperbolas and hyperboloids can be generated based on the generation of ellipses and ellipsoids by a simple transformation. The methods suggested in this paper are much simpler and easier to apply than those given by Eugenia Montiel et al.[2]

### 1.1 Curve Subdivision Scheme

G. Morin, J. Warren et al. described a simple and efficient non-stationary subdivision scheme of order four to generate circle and surface of resolution[1]. This curve scheme unifies known subdivision rules for cubic B-splines, splines-in-tension and a certain class of trigonometric splines capable of reproducing circles. In the following we firstly introduce the scheme(We call it *M-W* scheme) for close meshes.

Given a uniform knot sequence $T_k$ of the form

$$\beta_k\mathbb{Z}=\{\cdots,-2\beta_k,-\beta_k,0,\beta_k,2\beta_k,\cdots\}$$

the knot spacing between successive level is related by a factor of two, i. e., $\beta_k=\frac{1}{2}\beta_{k-1}$. Since the initial knot sequence $T_0$ is centered at the origin, the knot sequence $T_{k-1}$ is a subsequence of $T_k$. The subdivision mask associated with knot sequence $T_k$ is

$$s_{k-1}=\frac{1}{4+4\alpha_k}(1,2+2\alpha_k,2+4\alpha_k,2+2\alpha_k,1) \tag{1.1}$$

where $\alpha_k(\alpha_k\geq-1)$ satisfies

$$\alpha_k=\sqrt{\frac{1+\alpha_{k-1}}{2}}. \tag{1.2}$$

**Theorem 1.1**[1] *Given* $\beta_0=\frac{2\pi}{m}(m>2)$, $\alpha_0=\cos[\beta_0]$ *and a regular m-gon as the initial control polygon, the resulting curve generated by the M-W is a circle.*

From Theorem 1.1, if we choose $\{(\frac{\pi}{2}\cos(\frac{\pi}{2}k),\frac{\pi}{2}\sin(\frac{\pi}{2}k))\}_{k\in\mathbb{Z}}$ as the initial control points (a square) and $\alpha_0=\cos(\frac{2\pi}{4})=0$, then we get a unit circle by using the *M-W* scheme. Thus we have:

**Corollary 1.2** *Given knot sequence* $T_0=\frac{\pi}{2}Z$ *and*

$\{\frac{\pi}{2}(k,\cos(\frac{\pi}{2}k))\}_{k\in Z}$ *as initial control points, the resulting curve generated by M-W scheme is* cos $[x]$.

Figure 1: cos[x]

**Corollary 1. 3** *Given knot sequence* $T_0=\frac{\pi}{2}Z$ *and* $\{\frac{\pi}{2}(k,\sin(\frac{\pi}{2}k))\}_{k\in Z}$ *as the initial control points, the resulting curve generated by the M-W scheme is* sin $[x]$.

## 2 Generation of Conics

Based on Corollary 1. 2 and 1. 3, we will present methods to generate conics in this section.

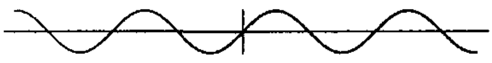

Figure 2 sin[x]

### 2. 1 Generation of ellipses

Let

$$f(\theta)=(a\cos\theta, b\sin\theta), \theta\in[0,2\pi], a>0, b>0. \tag{2.1}$$

be the parametric equation of an ellipse. According to Corollary 1. 2 and 1. 3, if we choose $\alpha_0=0$ and the initial control points as $\{\frac{\pi}{2}(a\cos(\frac{\pi}{2}k), b\sin(\frac{\pi}{2}k))\}_{k\in Z}$ (a diamond), the resulting curve generated by *M-W* scheme is an ellipse satisfies equation (2. 1). However, the resulting ellipse is not tangent to the initial control polygon. In the following, we modifies the above scheme to generate an ellipse which is tangent to the four sides of a given rectangle (which is the initial control polygon).

Given an arbitrary rectangle, suppose the length and width of the rectan-gle is $2a, 2b$ respectively. Choose the center of the rectangle as the origin, the lines connecting the midpoints of the opposite sides of the rectangle as two axes. We thus set up an orthogonal coordinate system. Connecting the midpoints of the four sides of the rectangle in turn, zoom in this diamond by a factor $\frac{\pi}{2}$, we get a control polygon with verties $\{\frac{\pi}{2}(a\cos(\frac{\pi}{2}k), b\sin(\frac{\pi}{2}k))\}_{k\in Z}$, Using *M-W* scheme, the resulting curve is an ellipse which is tangent to the given rectangle, see Figure 3.

### 2. 2 Generation of hyperbolas

The parameterized form of a hyperbola is

$$f(\theta)=(a\sec\theta, b\tan\theta), \theta\in[0,2\pi], a>0, b>0. \tag{2.2}$$

Since $\sec\theta=\frac{1}{\cos\theta}, \tan\theta=\frac{\sin\theta}{\cos\theta}$, from (2. 1) and (2. 2), we can translate a part of an ellipse into a part of hyperbola by a simple transformation, see Figure 4.

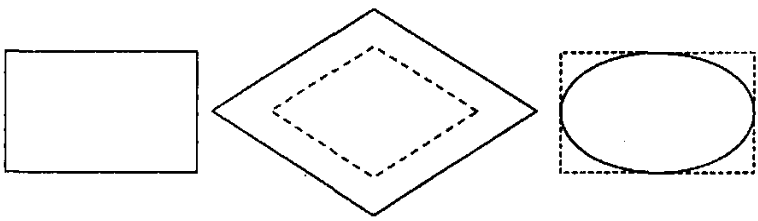

Figure 3: Left is the given rectangle; The dashed diamond in the middle is formed by connecting the four midpoints of the rectangle, the solid diamond is the inflated diamond; The dashed rectangle in the right is the given rectangle (the left one), the solid curve is the ellipse obtained by M-W subdivision scheme.

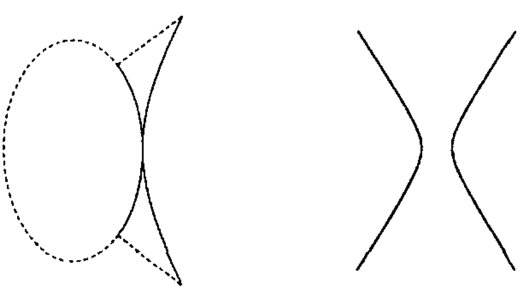

Figure 4: The left figure shows how to map a part of an ellipse to a part of a hyperbola with the same $a=2$ and $b=3$. The right figure shows the hyperbola obtained by the transformation.

## 3 Generation of Quadrics

In this section, we extend the idea for generating conics in the last section to generating quadrics.

### 3. 1 Generation of ellipsoids

The equation of an ellipsoid in parameterized form is

$$f(\theta,\phi)=(a\sin\theta\cos\phi, b\sin\theta\sin\phi, c\cos\theta),$$
$$\theta\in[0,\pi], \phi\in[0,2\pi], a,b,c>0. \tag{3.1}$$

We use the concept of longitude and latitude to simplify the process of generation of ellipsoids. We call the control polygon in a plane containning $Z$ axis *longitude control polygon*, and the control polygon in a plane which is tangent to $Z$ axis *latitude control polygon*.

Given an arbitrary cubic as the control mesh, we want to get an ellipsoid which is tangent to the cubic. Suppose the length, width and height of the given cubic are $2a$, $2b$, $2c$ respectively. Choose the center of the cubic as the origin, the lines connecting the center points of the opposite faces as the three axes, we thus get an orthogonal coordinate system. Connecting the centers of each face of the cubic to get an octahedron. Now scale the $X$ and $Y$ coordinates of the octahedron by a factor of $(\frac{\pi}{2})^2$ and the $Z$ coordinate by a factor of $\frac{\pi}{2}$, then subdivide the longitude control polygons and latitude control polygons using $M$-$W$ scheme respectively, the resulting surface is an ellipsoid, see Figure 5.

**Remark 3.1** *The process above can be seen as firstly subdivide the longitude control polygons and then subdivide latitude control polygons. In the first step we get two ellipses* $\frac{x^2}{(\frac{\pi}{2}a)^2}+\frac{z^2}{c^2}=1$ *and* $\frac{y^2}{(\frac{\pi}{2}b)^2}+\frac{z^2}{c^2}=1$. *In the second step we subdivide latitude control polygons which are diamonds with control points* $\{(\sqrt{1-\frac{z^2}{c^2}}\frac{\pi}{2}\cos(\frac{\pi}{2}k), \sqrt{1-\frac{z^2}{c^2}}\frac{\pi}{2}\sin(\frac{\pi}{2}k), z)\}_{k\in Z}$ *to get an ellipse* $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1-\frac{z^2}{c^2}$. *The result surface is an ellipsoid which is tangent to the given cubic.*

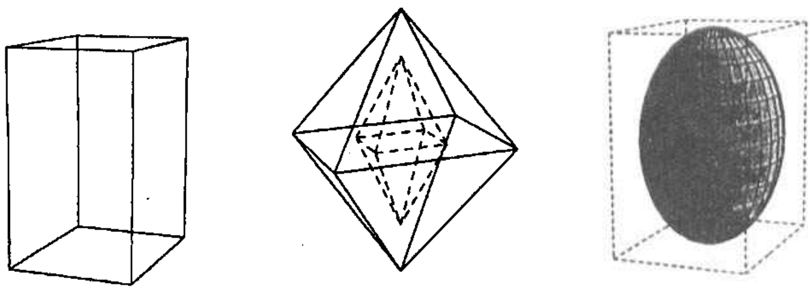

Figure 5: Left is the given cubic; The dashed octahedron is obtained by connecting the centers of the faces of the cubic; The solid octahedron is the inflated from origin Octahedron; The dashed cubic is the given cubic (the left one) and the solid shape is the ellipsoid.

### 3.2 Generation of hyperboloid

The parameterised form of single hyperboloid is

$$f(\theta,\phi)=(a\sec\theta\cos\phi, b\sec\theta\sin\phi, c\tan\theta),$$
$$\theta\in(-\frac{\pi}{2},\frac{\pi}{2}), \phi\in[0,2\pi], a,b,c>0. \quad (3.2)$$

The parameterised form of double hyperboloid is

$$f(\theta,\phi)=(a\tan\theta\cos\phi, b\tan\theta\sin\phi, c\sec\theta),$$
$$\theta\in(0,\frac{\pi}{2})\cup(\frac{\pi}{2},\pi), \phi\in[0,2\pi], a,bc>0. \quad (3.3)$$

The generation of the single hyperboloid and double hyperboloid is similar, so we only discuss the subdivision generation of single hyperboloid in the following.

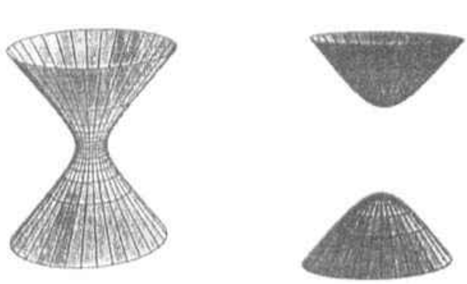

Figure 6: Left is a single hyperboloid. Right is a double hyperboloid.

We set up an orthogonal coordinate system and get the control octahedron as in section 3.1. We subdivide the longitude control polygon and get two ellipses and map the two ellipses to hyperbolas $\frac{x^2}{(\frac{\pi}{2}a)^2}-\frac{z^2}{c^2}=1$ and $\frac{y^2}{(\frac{\pi}{2}b)^2}-\frac{z^2}{c^2}=1$ as in section 2.

2. Then we subdivide latitude control polygons and get the single hyperboloid $\frac{x^2}{(\frac{\pi}{2}a)^2}+\frac{y^2}{(\frac{\pi}{2}b)^2}-\frac{z^2}{c^2}=1$, see Figure 6.

## 4 Examples

In this section, we will show some examples to generate ellipse-tori and ellipse-base-cylinder using the methods developed in the last section.

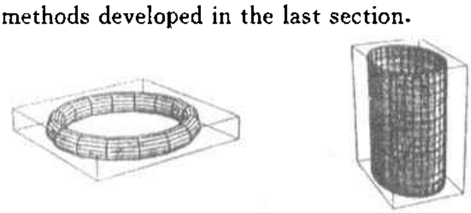

Figure 7: Left is an ellipse-tori obtained by revolve an ellipse along another ellipse. Right is an ellipse-base-cylinder which obtained by replace the circle with an ellipse as the base.

## 5 Conclusion and Future work

In this paper, we presented methods to generate conics and quadrics based on subdivision scheme

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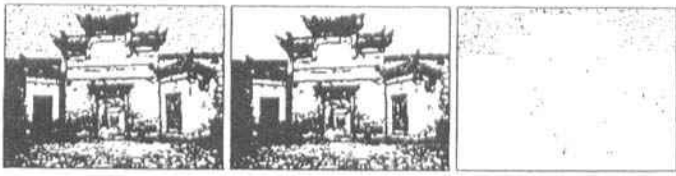
该组实验在没有人为介入的情况下由计算机自动完成。结果表明该算法具有较高的处理效率。

其次，对图1.a)所示的图像在不同阈值下进行滤噪修正实验。在不同的阈值下，本文算法取得了不同的滤噪效果。在所有的情况下，图中分散的色阶差别显著的噪声点均被有效去除，三个块状体的变化十分细微而不易察觉。实验表明，在适当的阈值范围内，检测的噪声点数不同仅仅反映了算法对色阶变化的灵敏程度。

图3是对一幅二值图像的滤噪结果。在对二值图像进行噪声过滤时，由于本文提出的算法主要关注图像中黑色像素稀疏区域，因此它可以在较好地保持图像主体的同时，高效地去除分散于图像中的杂质。该算法已成功地用于集成地对大规模扫描文本图像和图纸集的噪声过滤，在有效地提高图像视觉效果的同时大大减轻了图像处理的工作量。



a) b) c)

图3 二值图像的滤噪结果。a)原始图像 b)滤噪后图像 c)滤除的像素

**结论** 图形图像滤噪处理具有广泛的应用领域。然而，基于不同的应用领域，对图像噪声的认识不同，因此可以得到不同的定义。本文基于聚类的思想，将图像噪声视为少量的明显区别于周围区域的像素点集，并通过严格的数学方法加以定义和算法构造。其算法思想从根本上区别于其它的基于小波变换的方法，是聚类方法在图形图像处理领域应用的有益尝试。探讨基于上述思想的视频流噪声过滤以及图形图像轮廓提取与分类等是后续工作的研究内容。

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developed by G. Morin, J. Warren et al. in[1]. Given the control polygon (polyhedron), the corresponding ellipse(ellipsoid) is generated conveniently. The hyperbola and hyperboloid can be generated based on the generation of ellipse and ellipsoid by simple transformation. The methods developed in this paper are much simpler and easier to apply that those given by Eugenia Montiel et al.[2]. In the future, we will consider whether we can use M-W scheme to generate other curves or surfaces that have trigonometric forms.

## 6 Acknowledgement

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